Coordinate Transformation Approach to the Solution of the Fabry-Perot Open Resonator

T. Karpisz, B. Salski, P. Kopyt
Institute of Radioelectronics and Multimedia Technology
Warsaw University of Technology
Warsaw, Poland
tkarpisz@ire.pw.edu.pl

J. Krupka
Institute of Microelectronics and Optoelectronics
Warsaw University of Technology
Warsaw, Poland

***Abstract*—A new technique based on coordinate transformation dedicated to the analysis of an open Fabry-Perot resonator is introduced in this paper. The method allows reducing the problem to the scalar one-dimensional analysis of the resonator consisting of two Gaussian mirrors, which fit better to a Gaussian mode than spherical mirrors commonly used in that kind of open resonators. The proposed method is validated rigorously in the 20-40 GHz frequency range using a finite difference time domain method.**

***Index Terms*—Fabry-Perot open resonator, measurement techniques, FDTD.**

I. Introduction

Fabry-Perot open resonator (FPOR) is widely used in microwave frequency range due to the outstanding level of a quality factor exceeding well over 10^5, so it can be applicable to the measurement of a loss tangent of a material under test (MUT) as low as 10^{-5} [1-3]. So far, the best method of solving FPOR with the MUT has been based on a characteristic equation derived assuming that the modes are of Gaussian type, while mirrors are spherical [1 - 3]. However, that method suffers from limited accuracy resulting from approximate approach to the problem and, apparently, from the mismatch between the mode Gaussian profile and mirrors spherical curvature.

Those reasons have encouraged the authors to investigate new solutions of the FPOR, which may be more accurate than state-of-the art in that subject. The main family of TEM modes in the FPOR is axisymmetrical and they have a Gaussian shape at longitudinal and transverse cross-sections. As it will be shown, coordinate transformation techniques allow reducing the problem from vector three-dimensional (3D) to the scalar 1D layered problem.

II. FPOR representation

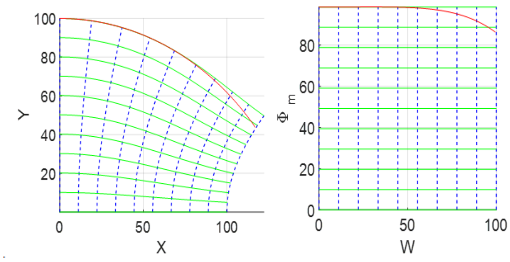
Electric field of a fundamental $TEM_{0,0,q}$ mode is given as follows [4]:

$$E(\varrho, y) = E_0 \frac{w_0}{w(y)} exp\left(\frac{-(x^2+z^2)}{w(y)^2}\right) exp(-j\Phi_m \mathrm{k}) \tag{1}$$

where

$$\Phi_m = y + \frac{x^2+z^2}{2R} - k^{-1}\,\mathrm{atan}\left(\frac{y}{y_R}\right) \tag{2}$$

represents a wavefront, k is wavenumber, R is the radius of the wavefront given in [5, Eq. (4.7.17b)] and y_r stands for the Rayleigh range in [5, Eq. (4.7.16)].



(a) (b)

Fig. 1. Gaussian wavefronts (green line) and field lines (blue line) in (a) Cartesian and (b) Gaussian coordinate systems. Red line indicates a spherical Fabry-Perot mirror (R_0 = 100 mm).

According to the authors knowledge, most (if not all) known FPOR realizations uses spherical mirrors, which is counter-intuitive as the wavefront is Gaussian. Since only $TEM_{0,0,q}$ modes are investigated, it has been assumed that electric field lines (W = const) are orthogonal to the wavefronts (Φ_m = const) which are tangential to electric and magnetic field components:

$$W = \frac{\varrho}{\sqrt{1+\left(\frac{y}{y_R}\right)^2}} \tag{3}$$

while the azimuthal coordinate can be represented as:

$$\varphi = \mathrm{atan}\left(\frac{z}{x}\right) \tag{4}$$

Equations (2), (3) and (4) allow introducing a new orthogonal Gaussian coordinate system (GCS). In GCS, Gaussian mirrors and wavefronts become planar (Fig.1). As it can be noticed in (1), (2), the proposed coordinate system is frequency dependent. However, frequency variation of the coordinates is at the level of a few μm/GHz at a distance where the mirrors are placed, which means negligible impact on the accuracy of the method in a broad spectrum.

As shown in [6], Maxwell equations can be kept in the same form after coordinate transformation provided that medium properties are appropriately transformed. For that purpose, coordinate transformation Jacobian tensor needs to be calculated, which can be used to calculate complex permittivity and permeability tensors, as shown in [6]. 3D GCS problem (Φ_m, W, φ) can be further reduced to the 2D one (Φ_m, W) if azimuthal permittivity and permeability tensor components are scaled as $n''_{33} = n'_{33}/W$, and lateral components as $n''_{11} = n'_{11}W$ and $n''_{22} = n'_{22}W$ [7], where n denotes either ε or μ. Assuming that electric (magnetic) field is W-(φ–)polarized, ε''_{11} (μ''_{22}) component represents scalar permittivity (permeability) of a newly defined inhomogeneous medium.

The problem can be further reduced to the Φ_m-oriented 1D transmission line problem with effective permittivity and permeability at a given Φ_m coordinate calculated as follows:

$$\varepsilon_{eff}(\Phi_m) = \frac{\int_0^\infty \varepsilon_r(\Phi_m, W)\,\varepsilon''_{22}(\Phi_m)\,E(\Phi_m)\,dW}{\int_0^\infty E(\Phi_m)\,dW} \quad (5)$$

where $\varepsilon_r(\Phi_m, W)$ stands for bulk relative permittivity at a certain transverse cross-section of the FPOR (e.g. that of MUT or air), and electric field $E(\Phi_m)$ is assumed to be a priori given, as in (1). Effective permeability $\mu_{eff}(\Phi_m)$ shall be calculated in a similar manner.

III. FPOR Calculation

A 1D FPOR problem can be solved by splitting the cavity in half [8] and calculating S_{11} of one half of the corresponding 1D stratified transmission line [9], which leads to the spectrum of wave admittance at a splitting plane. It should be stressed out that a flat sample of MUT becomes concave after transformation to GCS. However, that fact can be accounted for in (5) by including correct medium properties, $\varepsilon_r(\Phi_m, W)$ and $\mu_r(\Phi_m, W)$, while calculating effective properties of the equivalent 1D transmission line at a given Φ_m coordinate.

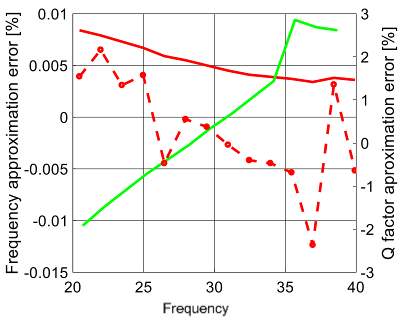


Fig. 2. Frequency approximation error of empty resonator (solid green line), resonator with PTFE sample (solid red line) and for the right axis Q-factor of resonator with PTFE sample (red dashed line).

Resonant frequencies are indicated by the points, where $Y''_{tot} = 0$, while the total Q-factor of the FPOR can be calculated as described in [10]. Validation of the proposed method has been undertaken with the aid of a commercially available software, QuickWave-V2D, based on a vector-2D finite difference time domain method (FDTD) algorithm formulated in cylindrical coordinates [11]. Magnetic symmetry has been enforced at $W = 0$ since odd modes are considered only. An FDTD cell size has been set to be a = 75 μm, which means 100 FDTD cells per wavelength at 40 GHz in vacuum.

Figure 2 presents resonant frequencies and the Q-factors of odd $TE_{0,0,q}$ modes have been shown. An empty and PTFE-filled FPOR have been analysed [12]. It has been assumed that PTFE is 4 mm thick and has the following properties: ε_r = 2.04, $\tan\delta = 1.135\times10^{-5}$. A maximum frequency error reaches 8 MHz, which is at the same level as a numerical dispersion error of FDTD (ca. 0.02% in free space) [13]. The reference Q-factor obtained from FDTD simulations is also loaded with some uncertainty, as the simulation of the problems of that high Q-factor is very challenging. Further experimental validation needs to be undertaken to verify if the real error is even smaller.

IV. Conclusion

A novel approach to the analysis of the Fabry-Perot open resonator is presented in this paper. The method is based on coordinate transformation, which allows reducing a vector 3D problem to a scalar 1D one. The proposed method has been validated against rigorous FDTD simulations and promising results have been obtained. Further development of the method and experimental analysis are needed to confirm the usefulness of the approach to the characterization of materials in microwave frequency spectrum and beyond.

Acknowledgement

This work was partly funded from the project “High-precision techniques of millimeter and sub-THz band characterization of materials for microelectronics” operated within the Foundation for Polish Science TEAM TECH Programme co-financed by the European Regional Development Fund, Operational Programme Smart Growth 2014-2020.

References

[1] A. L. Cullen and P. K. Yu, “The accurate measurement of permittivity by means of an open resonator,” *Proc. Royal Soc. London,* vol. A325, 1971, pp. 493-509.

[2] A. L. Cullen and P. K. Yu, “Measurement of permittivity by means of an open resonator. I. Theoretical,” *Proc. Royal Soc. London*, vol. A380, 1982, pp. 49-71.

[3] T. M. Hirvonen, P. Vainikainen, A. Lozowski, and A. V. Raisanen, “Measurement of Dielectrics at 100 GHz with an Open Resonator Con4, 1996, pp. 780-786.

[4] R. G. Jones, “Precise dielectric measurements at 35 GHz using an open microwave resonator,” *Proc. IEE*, vol. 123, no. 4, April 1976.

[5] O. Svelto, *Principles of Lasers*. New York, NY, USA: Springer, 2010.

[6] C. Kottke, A. Farjadpour, and S. G. Johnson, “Perturbation theory for anisotropic dielectric interfaces, and application to sub-pixel smoothing of discretized numerical methods,” *Phys. Rev. E*, vol. 77, p. 036611, 2008.

[7] M. Celuch and W. K. Gwarek, “Industrial Design of Axisymmetrical Devices Using a Customized FDTD Solver from RF to Optical Frequency Bands”, *IEEE Microwave Magazine,* vol. 9, pp. 150-159, Dec 2008.

[8] P. Kopyt, B. Salski, and M. Sakowicz, "Efficient 3D electromagnetic modeling of metal-metal waveguides employed for quantum cascade lasers operating in the THz band," *J. Lightw. Technol.*, early access, 2018.

[9] S. J. Orfanidis, *Electromagnetic Waves and Antennas*, 2013, [online] Available: http://eceweb1.rutgers.edu/~orfanidi/ewa/.

[10] M. Gustafsson, D. Tayli, and M. Cismasu, "Q factors for antennas in dispersive media," arXiv:1408.6834v3, Dec. 2014.

[11] QuickWave-V2D, QWED Sp. z o.o., 1997–20178. [Online]. Available: http://www.qwed.com.pl.

[12] J. Krupka, "Measurements of the Complex Permittivity of Low Loss Polymers at Frequency Range From 5 GHz to 50 GHz," *IEEE Microw. Compon. Lett.*, vol. 26, no. 6, pp. 464-466, June 2016.

[13] A. Taflove and S. Hagness, *Computational Electromagnetics- The Finite-Difference Time-Domain Method*, 3rd Edition. Boston-London: Artech House, 2005.